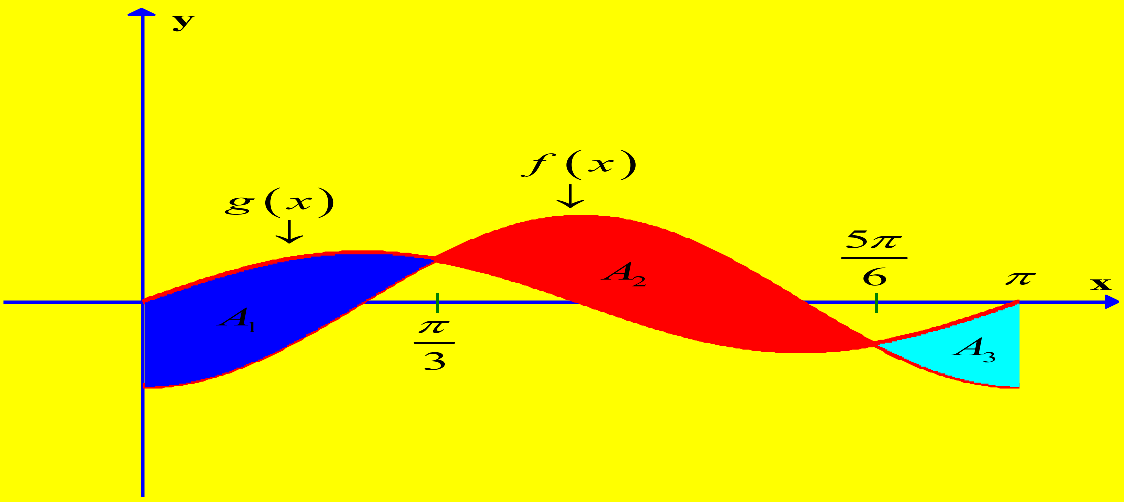
Concept based and Exam oriented

TUTORIAL LESSONS

Mathematical Methods

3 & 4



In This Book

Fully worked solutions to all questions

Written explanation of theory and concepts for each question

Over 1000 specifically designed study and exam style questions

Strong focus on algebra from basic to the highest complexity

Answers to most of your questions

Help for homework and assessment

Exam preparation through each lesson

Systematic revision while study new topics

B. Z.

0405 213 375

About the book

Even though this book is fully compatible with the Victorian school Curriculum and contains all important topics and can also be very useful as supplementary study material in other states, the book is still different than any school text book as well as from most of the other mathematics books because all questions in this book are fully worked out and concisely explained in full details.

This book is designed to help you to find answers to, almost, all of your questions. Written by tutor with over 30 years tutoring experience, the book contains over 1000 fully worked questions with concise theoretical and conceptual explanation for each question. The intellectual base for this book is a systematic collection of questions asked by students over a long period of time. In order to provide students with all subject requirements the author created questions by adding other mathematical aspects such as: plenty algebra, application of mathematical theory and development of mathematical concepts and all this with strong tests and exam orientation. Finally, the wide range of different style of questions is helping students to learn mathematics with deeper understanding of theory and concepts that can be applied across the whole range of questions, rather than trying to apply some memorized patterns that usually can work on some certain questions. So this book will, definitely, help you:

- To find answers to most of your questions (even the most complex once)
- To do your homework and school tasks with easy and satisfactory way
- To be ready for your school assessments and final year exam
- To improve algebra skills
- To develop independent studying skills
- To get knowledge of the highest standard in the fastest time
- To understand and memorise even the most complex concepts in an easy way because most of the concepts are broken into simple steps and constantly revised throughout the book.
- To develop a high ability to apply theoretical knowledge and concepts in solving questions.
- To extend your educational ability and to increase motivation as you will discover that “knowledge is the power” and studying is not such hard work.

The book contains all important topics for Mathematical Methods 3&4, such as:

- Functions, Relations and Transformation of Graphs
- Exponential and Logarithmic Equations and Functions
- Trigonometry
- Differentiation and Application of Differentiation
- Integration and Application of Integration

There are 35 tutorial lessons. Each lesson (session), contains in average about 25 fully worked out questions with concise explanation of theory and concept applied for each question. The firs (or first few) sessions of each topic were designed to present theory and basic concepts for that topic. The other sessions are practices sessions giving to students the best examples to get full understanding of the subject. Questions are not arranged according to their level of complexity, because all questions are fully worked out and concisely explained so that every student (regardless of current level of knowledge) can easily understand algebra, concept and theory applied on each questions. The other reason why questions are not graded is to motivate students to work across the entire book rather than to work selectively. Such arrangement and very detailed explanation of questions enable to students of all levels of knowledge to progress with this book to the highest possible standard.

Finally, this book is an attempt to save student's time and money seeing tutors around and to make available to everyone the best tutorial lessons that has been delivered to students over a few decades.

A short author’s message about how to study Mathematics

Choosing the right way of doing something is very solid base for success. As studying is a complex activity, so we have to be more conscious of choosing the right way of studying. This way is, generally, different from subject to subject and as Mathematics is one of very specific and quite complex subject, we have to take the right approach to that subject to ensure our success.

Definitely, we can’t study Mathematics by reading even not by memorising simple facts and formulas. Studying Mathematics requires permanent practice over long period. As we are progressing from grade to grade, we have to keep fresh our previous knowledge (theory, concepts, even facts and formulas) and to build up that knowledge from current program.

When we start studying a new chapter, of course, we have to “read” the book. During “reading” time we have to stop “reading” whenever is necessary to write down important words, facts and formulas or to draw a graph or some other appropriate drawing, even if that graph or drawing is already in the book. During that time we have to identify the meaning of new terminology, maybe some formulas and definitions, key facts, theoretical explanations, probably some new concepts and finally to identify previously learned knowledge we should apply in the current chapter. When you are happy with your understanding of what you “read” then you can move to the next phase of studying. That is practicing. But how someone can be sure that his/her understanding is good enough to move to practicing? Here I wouldn’t suggest any quiz or simple test questions trial. Instead, I would recommend a different technique. That is to visualise in front of you your friend, cousin or your parents and try to teach them of what you have been learning during “reading” time. If necessary, go back in book to refresh your memory or to check if you have missed something. The theory of each chapter of most of the books is usually written on just a few pages, so that “reading” process shouldn’t consume long time. One or two hours should qualify you to move to practicing phase.

Once when you start practicing, you can start from easier questions and go to the most complex once. For each question, before you start working, spend few minutes time to think about an appropriate concept and draw a simple strategy how to apply that concept. When you start work make sure that your work is inside your concept and strategy drawn at the beginning. If the concept, you have, doesn’t work for that question you should try to find out the clear reason why that concept is not good. That will help you to correct the concept and satisfactory to finish the question.

When due to move on another chapter, you should select some testing questions and to conduct short test of your understanding of current chapter. It is very useful, at the end of studying the chapter, once again, pretend to be a teacher and try to teach somebody. Compare your way of “teaching” somebody after “reading” phase and after practising phase of the end of studying the chapter and you will discover how big progress you made.

How to use this book

Thinking about method of how to study Mathematics, you already may have some ideas how to use this book. Even though you can use this book independently from any other books, still it would be better to study the chapter in your school and when you already have some knowledge of the chapter then to start using this book. Go over first session of the chapter to establish solid understanding of theory and important concepts. Even though you have all questions fully worked out, I am suggesting you to try to solve questions before checking working solution. After each question even if you solve the question correctly, go through worked solution with high attention as always there will be something there that you may learn. It is, also, recommended when going through worked solutions, to frequently reefer to theoretically session(s) to learn how the author is applying theory and concepts on practical questions.

Let start

Good luck

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Integration and Application of Integration................................385

S E S S I O N 2

FUNCTION, RELATIONS
&
TRANSFORMATION OF GRAPHS

Further Definitions, Terminology, Concepts and Study Examples

Study Worksheet – Session 2

1) For the following functions/relations
   a) State the largest possible domain that inverse will exists
   b) State the domain and range of inverse function.
   c) Find expression for inverse function.
   d) Sketch both $f(x)$ and $f^{-1}(x)$ on the same set of axes.
   e) Find intersection points between $f(x)$ and $f^{-1}(x)$

1)	$y = 2x + 1$	2)	$y = 1 + \dfrac{2}{x+4}$	3)	$y = \sqrt{x-3} + 1$
4)	$y = \dfrac{1}{\left(x + \frac{1}{2}\right)^2} - 3$	5)	$y = 2x^2 + 3x - 2$	6)	$y = 3 - 2\sqrt{x-4}$
7)	$y = x^2 - x + 3$	8)	$y = 3 - \dfrac{1}{(x+12)^2}$		

Key concepts in solving questions involving inverse functions

a) Only one-to-one function may have inverse function. If function is not one-to-one, we have to restrict its domain so that we will have one-to-one function. Usually (if no other instructions), restriction of domain going through turning point or vertical asymptote.

b) Domain of inverse function is a range of original function, and the range of inverse is domain of original function.

c) Expression for inverse function can be obtained from expression of original function by swapping x and y variables in original expression and transposing that expression to make y subject. This usually requires algebra skills needed for solving equations.

d) Line $y = x$ is a line of symmetry between original and inverse graph. It is helpful to use this line when sketching inverse function graph (especially if inverse function is not one of standard functions we learned).

e) If graphs of original and inverse functions intersect each other, then their intersection point(s) must be on the line $y = x$. This fact can be used when looking for coordinates of intersection point(s) between original and inverse graph, by setting any (always chose simpler) of the following three equations:
$f(x) = f^{-1}(x)$ or $f(x) = x$ or $f^{-1}(x) = x$

Note: The concepts above, generally, can be used for any question involving inverse functions. The order of concept stated above is matching the order in question below just for easier application.

Solutions to Study Worksheet – Session 2

1.

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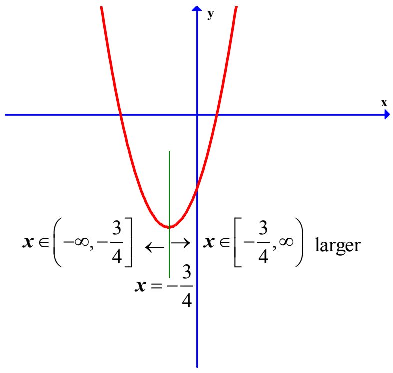
5. a) $y = 2x^2 + 3x - 2$

Parabola is also, many to one function and it doesn't have inverse function over its implied domain. It is necessary to restrict domain, so that part of parabola will have inverse function. Restriction of domain, if not stated differently, should go through turning point. So we have to find coordinates of turning point.



$y = 2x^2 + 3x - 2$

$y = 2\left(x^2 + \frac{3}{2}x - 1\right)$

$y = 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} - 1\right]$

$y = 2\left(x + \frac{3}{4}\right)^2 - \frac{25}{8}$

$T\left(-\frac{3}{4}, -\frac{25}{8}\right)$ turning point

We need x coordinate of turning point. In this case it is $x = -\frac{3}{4}$ and than to chose larger set of x values for domain of original function that inverse will exist. So subset $\left[-\frac{3}{4}, \infty\right)$ is a domain of $y = 2x^2 + 3x - 2$ that inverse will exist.

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b) Domain of inverse = range of original That is $x \in \left[-\frac{25}{8}, \infty\right)$

Range of inverse = domain of original That is $\left[-\frac{3}{4}, \infty\right)$

c) $y = 2x^2 + 3x - 2$

$x = 2y^2 + 3y - 2$

We can choose two different ways to make y subject in this equation; by completing the square or applying quadratic formula. Student should understand both ways, but always choose easier way whenever is possible.

Completing square

$2\left(y^2 + \frac{3}{2}y - 1\right) = x$

$y^2 + \frac{3}{2}y - 1 = \frac{x}{2}$

$\left(y + \frac{3}{4}\right)^2 - \frac{9}{16} - 1 = \frac{x}{2}$

$\left(y + \frac{3}{4}\right)^2 = \frac{x}{2} + \frac{25}{16}$

$y = \frac{-3 \pm \sqrt{8x + 25}}{4} = f^{-1}(x)$

Applying quadratic formula

$x = 2y^2 + 3y - 2$

$2y^2 + 3y - 2 - x = 0$

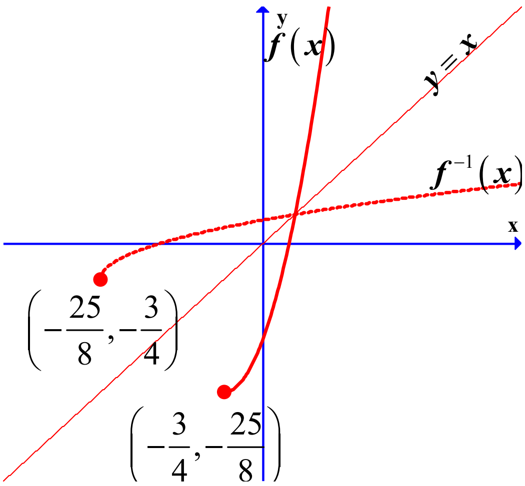
$y = \frac{-3 \pm \sqrt{9 + 8(2 + x)}}{4}$

$y = \frac{-3 \pm \sqrt{25 + 8x}}{4} = f^{-1}(x)$

We have the same equation and in this example it is easier to apply formula, but student should understand both way

As we are working only with positive part of parabola, then it will be "+" instead of "$\pm$" in expression above and we can write expression for inverse function as $y = \frac{-3+\sqrt{8x+25}}{4} = f^{-1}(x)$

d)



Note: this is square root function and its standard form is:

$$y = -\frac{3}{4} + \frac{\sqrt{8\left(x+\frac{25}{8}\right)}}{4}$$
$$y = -\frac{3}{4} + \frac{2\sqrt{2}}{4} \times \sqrt{x+\frac{25}{8}}$$
$$y = -\frac{3}{4} + \frac{\sqrt{2}}{2} \times \sqrt{x+\frac{25}{8}}$$

e)

$$f(x) = x$$
$$2x^2 + 3x - 2 = x$$
$$2x^2 + 2x - 2 = 0$$
$$x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1+4}}{2}$$

We have to choose only positive solution as negative solution is out of domain. We have to keep in mind that we have restricted domain so that inverse function can exist.

$$x = \frac{-1+\sqrt{5}}{2}$$

$\left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$ Coordinates of intersection point. It is not necessary to calculate y coordinate as we have line $y = x$ through that point. So x and y coordinate of intersection point(s) between original and inverse graphs ar always the same.

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S E S S I O N   12

EXPONENTIAL & LOGARITHM EQUATIONS AND GRAPHS

SAC and Examination Style Questions

Session 12 – SAC and Exam Style Questions

1) Consider the function $f(x) = \ln(|x+2|) - 1$.

a) Find maximal domain
b) Find the exact coordinates of x and y intercepts.
c) Sketch the graph and indicate any asymptote with its equation.

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Session 12 – Solutions to SAC and Exam Style Questions

1) a) By the definition of logarithm $|x+2|$ must be positive. Then we will get domain by solving this inequation.

$$|x+2|>0 \quad \rightarrow \quad \begin{cases} x+2>0 & \Rightarrow \; x>-2 \\ -x-2>0 & \Rightarrow \; x<-2 \end{cases} \quad \Rightarrow \; R\setminus\{-2\}\text{domain}$$

(*Keep in mid definition of absolute value)*.

b) $f(x)=\ln(|x+2|)-1$

For x_{int} $\quad \ln(|x+2|)-1=0$

$$|x+2|=e \quad \rightarrow \quad \begin{cases} x+2=e \;\; for \;\; x+2>0 \\ x=-2+e \qquad x>-2 \\ and \\ -x-2=e \;\; for \;\; x+2<0 \\ x=-2-e \qquad x<-2 \end{cases}$$

So there are two *x* intersection points: $(-2+e,0)$ and $(-2-e,0)$

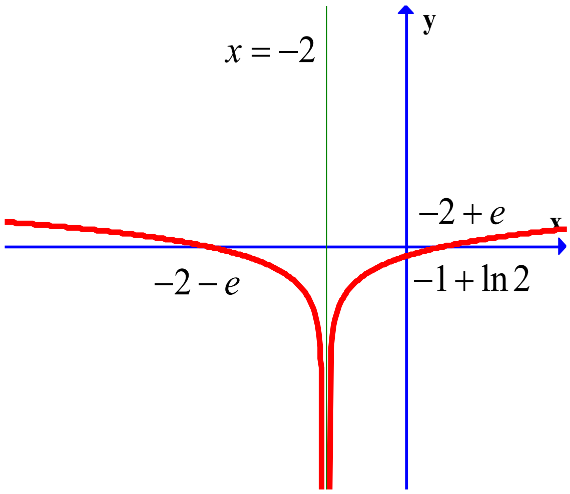
Note: When we solve absolute value equations, sometimes one or more solution(s) can be out of domain. Of course, we have to dismiss such solution(s).

For y_{int}, $x=0$ and $y=-1+\ln 2$

c) The best way to avoid mistake when sketching this type of graph is to change absolute value function to its hybrid form. **Note**: This function **is not** $f(x)=|\ln(x+2)|-1$

$$f(x)=\ln(|x+2|)-1=\begin{cases} f(x)=\ln(x+2)-1 \;\; for \;\; x>-2 \\ and \\ f(x)=\ln(-x-2)-1 \;\; for \;\; x<-2 \end{cases}$$

Now we can clear see that is only one vertical asymptote with equation $x=-2$ and we can sketch each function in its domain to obtain graph of $f(x)=\ln(|x+2|)-1$



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S E S S I O N   25

APPLICATION OF DIFFERENTIATION

Study Worksheet 2

Session 25 - Application of differentiation
Study Worksheet 2

1) An ice cube from a freezer is being placed in a room temperature so that ice cube is melting. The side length is decreasing at a rate of 0.2 cm/h. Calculate the rate at which the volume of ice cube is decreasing when side of cube is 6 cm.

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Session 25 - Application of differentiation
Solutions to Study Worksheet 2

Note: we know that relation between two variables is actually rule that telling us how change of one variable depends of change of other variable. Similar, the speed of change (rate) of one variable may depend of speed of change of the other variable. This, in mathematics, we recognize as **related rates** or relation between two rates. Mathematical problems involving relation between two rates generally can be solved by applying differentiation.

The concept of solving problems involving related rates is actually setting up one equation involving chain rule and then solving that equation. However, for easier understanding, we can break that concept in a few steps.

1. On left hand side of our equation, always should be rate (in respect with time) that we have to calculate.

2. On the right hand side should be rate of the same quantity but with respect of some other quantity (not with time) whose rate in respect with time is usually given in question.

Forming equation involving those two steps is actually writing a chain rule. Identifying given rate from text can be easier if we look for units in which that rate is given. For example; if given rate is in cm^2 / min then we have rate of area and similar.

1. Cube is a prism with length, width and height of the same length. It is obvious that if side length is increasing that volume will also increase. Question is asking to calculate the rate (speed) of volume decrease when we know the rate of side decrease. Related rates in this question are rate of volume $\left(\frac{dV}{dt}\right)$ and rate of side $\left(\frac{da}{dt}\right)$ decrease (we choose letter *a* to represent the length of side). On the other hand, since volume decreasing when side length decreasing then we have rate of volume in respect with side length $\left(\frac{dV}{da}\right)$. The chain rule is obvious as relation between those rates:

Rate of volume in respect with time = Rate of volume in respect with side × rate of side in respect with time

Mathematically written this statement is:

$$\frac{dV}{dt}=\frac{dV}{da}\times\frac{da}{dt}$$

We have to calculate $\frac{dV}{dt}$ (rate at which the volume is decreasing).
We know $\frac{da}{dt}=0.2$ cm/h (rate of side decreasing). To recognize this rate from text, sometimes it is useful to look units given (cm/h (=) length / time)
We need $\frac{dV}{da}$ (rate of volume in respect with side). To find this rate, we need formula (generally some equation) that connects volume and side and then to find derivative of V in respect to a. That is formula for volume of a cube.

$$V=a^3$$
$$\frac{dV}{da}=3a^2$$

Now substituting all in starting equation we have;

$$\frac{dV}{dt}=3a^2\times 0.2 \qquad \text{and for } a=6$$
$$\frac{dV}{dt}=3\times 6^2\times 0.2=21.6\,\text{cm}^3/\text{h}$$ Always write units in your final result.

The above concept and method could be applied on almost all questions of this sort (for this course) and that shouldn't be any issue. The starting equation (chain rule) is easy to set, even, if we have problem to understand text. Simply, on left side we can write rate that we need to calculate (in this question $\frac{dV}{dt}$) and on the other side of equation write the same rate split over two fractions with empty spaces in each fraction. For example for this question we could write:

$$\frac{dV}{dt}=\frac{dV}{\quad}\times\frac{\quad}{dt}$$

Now go back to text and concentrate only on given rate (in this question $\frac{da}{dt}$) and fill empty spaces with that rate. For this question write da above dt and dt below dV and we will get chain rule

$$\frac{dV}{dt}=\frac{dV}{da}\times\frac{da}{dt}$$

Note: the last suggestion is not any theoretical interpretation of related rates and also this is not any scientific approach, but it could be useful help to do some work in case when we can't understand text. Setting correctly chain rule will help us to get better understanding of text for our further work.

However, students must be careful when they need to use some formulas, or maybe to construct one (depending of question), and correctly to find needed derivative. This is very similar problem like we have with application of differentiation in solving minimum and maximum problems when we need to set "minor" equation.

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S E S S I O N   34

INTEGRATION - APPLICATION

Exam Style Examples 1

Session 34 – Application of Integration – Exam Style Examples 1

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13. Find the exact area bounded by the graphs of $f\left(x\right)=\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)$,

$g\left(x\right)=\cos\left(2x-\frac{\pi}{2}\right)$, the y axes and the line $x=\pi$

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Session 34 – Application of Integration – Solution to Exam Style Examples 1

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13. $f\left(x\right)=\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)$ $g\left(x\right)=\cos\left(2x-\frac{\pi}{2}\right)$ $x=\pi$

We have to sketch those two graphs. We can use calculator but we have to be careful as exact area requires (especially with *x* coordinates of their intersection point as we will use them for integration limits). In any case it is better to find intersection points (only x coordinates) without calculator, by solving an appropriate equation

$$f\left(x\right)=g\left(x\right)$$

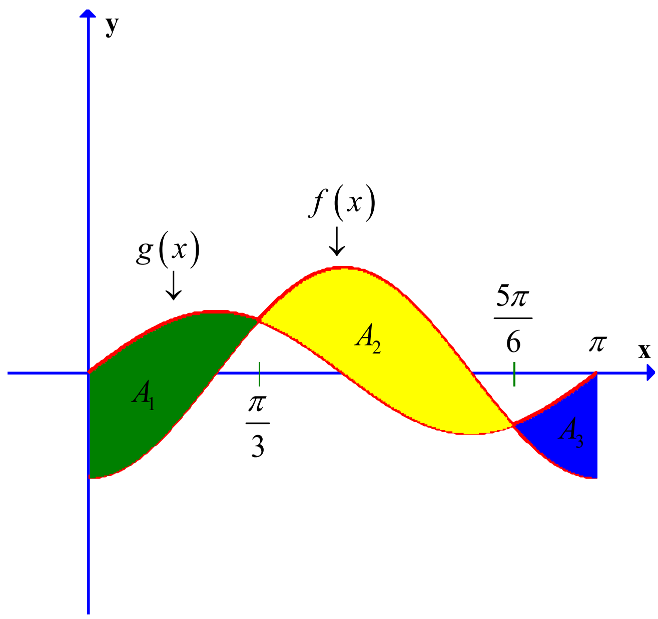
$$\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)=\cos\left(2x-\frac{\pi}{2}\right) \quad /\div\cos\left(2x-\frac{\pi}{2}\right)$$

$$\sqrt{3}\tan\left(2x-\frac{\pi}{2}\right)=1$$

$$\tan\left(2x-\frac{\pi}{2}\right)=\frac{1}{\sqrt{3}}$$

We have to adjust domain

$$0<x<\pi$$
$$0<2x<2\pi$$
$$-\frac{\pi}{2}<2x-\frac{\pi}{2}<2\pi-\frac{\pi}{2}$$
$$-\frac{\pi}{2}<2x-\frac{\pi}{2}<\frac{3\pi}{2}$$



Solutions are:

$$2x-\frac{\pi}{2}=\frac{\pi}{6},\frac{7\pi}{6}$$
$$2x=\frac{\pi}{6}+\frac{\pi}{2},\frac{7\pi}{6}+\frac{\pi}{2}$$
$$2x=\frac{2\pi}{3},\frac{5\pi}{3}$$
$$x=\frac{\pi}{3},\frac{5\pi}{6}$$

The total area is

$$A=A_1+A_2+A_3$$

$$A_1=\int_0^{\frac{\pi}{3}}\left[g(x)-f(x)\right]dx=\int_0^{\frac{\pi}{3}}\left[\cos\left(2x-\frac{\pi}{2}\right)-\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)\right]dx$$

$$A_1=\left[\frac{1}{2}\sin\left(2x-\frac{\pi}{2}\right)+\frac{\sqrt{3}}{2}\cos\left(2x-\frac{\pi}{2}\right)\right]_0^{\frac{\pi}{3}}$$

$$A_1=\left[\frac{1}{2}\sin\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)+\frac{\sqrt{3}}{2}\cos\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)\right]-\left[\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)+\frac{\sqrt{3}}{2}\cos\left(-\frac{\pi}{2}\right)\right]$$

$$A_1=\frac{1}{2}\sin\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cos\frac{\pi}{6}-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(-\frac{\pi}{2}\right)=\frac{1}{4}+\frac{3}{4}+\frac{1}{2}-0=\frac{3}{2}$$

$$A_2=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}}\left[f(x)-g(x)\right]dx=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}}\left[\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)-\cos\left(2x-\frac{\pi}{2}\right)\right]dx$$

$$A_2=\left[-\frac{\sqrt{3}}{2}\cos\left(2x-\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(2x-\frac{\pi}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{5\pi}{6}}$$

$$A_2=\left[-\frac{\sqrt{3}}{2}\cos\left(\frac{5\pi}{3}-\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(\frac{5\pi}{3}-\frac{\pi}{2}\right)\right]-\left[-\frac{\sqrt{3}}{2}\cos\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)\right]$$

$$A_2 = -\frac{\sqrt{3}}{2}\cos\frac{7\pi}{6} - \frac{1}{2}\sin\frac{7\pi}{6} + \frac{\sqrt{3}}{2}\cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{6} = \frac{3}{4} + \frac{1}{4} + \frac{3}{4} + \frac{1}{4} = 2$$

$$A_3 = \int_{\frac{5\pi}{6}}^{\pi} \left[g(x) - f(x)\right]dx = \int_{\frac{5\pi}{6}}^{\pi} \left[\cos\left(2x - \frac{\pi}{2}\right) - \sqrt{3}\sin\left(2x - \frac{\pi}{2}\right)\right]dx$$

$$A_3 = \left[\frac{1}{2}\sin\left(2x - \frac{\pi}{2}\right) + \frac{\sqrt{3}}{2}\cos\left(2x - \frac{\pi}{2}\right)\right]_{\frac{5\pi}{6}}^{\pi}$$

$$A_3 = \left[\frac{1}{2}\sin\left(2\pi - \frac{\pi}{2}\right) + \frac{\sqrt{3}}{2}\cos\left(2\pi - \frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\sin\left(\frac{5\pi}{3} - \frac{\pi}{2}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{5\pi}{3} - \frac{\pi}{2}\right)\right]$$

$$A_3 = \frac{1}{2}\sin\frac{3\pi}{2} + \frac{\sqrt{3}}{2}\cos\frac{3\pi}{2} - \frac{1}{2}\sin\frac{7\pi}{6} - \frac{\sqrt{3}}{2}\cos\frac{7\pi}{6} = -\frac{1}{2} + 0 + \frac{1}{4} + \frac{3}{4} = \frac{1}{2}$$

$A = \frac{3}{2} + 2 + \frac{1}{2} = 4$ sq. units

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